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Basin of Attraction of Cycles of Discretizations of Dynamical Systems with SRB Invariant Measures

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Computer simulations of dynamical systems are *discretizations*, where the finite space of machine arithmetic replaces continuum state spaces. So any trajectory of a discretized dynamical system is eventually periodic. Consequently, the dynamics of such computations are essentially determined by the cycles of the discretized map. This paper examines the statistical properties of the event that two trajectories generate the same cycle. Under the assumption that the original system has a Sinai–Ruelle–Bowen invariant measure, the statistics of the computed mapping are shown to be very close to those generated by a class of random graphs. Theoretical properties of this model successfully predict the outcome of computational experiments with the implemented dynamical systems.

KEY WORDS: Chaos; computation; collapse; computer arithmetic; computer artifact.

INTRODUCTION

There exists a rich qualitative theory of chaotic dynamical systems in terms of statistical properties like Sinai–Ruelle–Bowen (SRB) invariant measures.[27] Recall that the probability measure μ is called an SRB measure for f if there exists a neighbourhood U of the support Supp(μ) such that μ is a weak limit of the sequence of measures $\mu_n = (1/n) \sum_{i=0}^{n-1} f^i_* \delta_x$ for almost all initial conditions $x \in U$ with respect to Lebesgue measure. Here δ_x is the Dirac measure concentrated at x and f_* is the mapping in the space $\mathscr{P}(U)$ of Borel measures ν on U, defined by

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$f_*\mu(A)=\mu(f(A))$, where $\mu\in\mathscr{P}$ and A is a Borel subset of U. Such a measure describes the properties of exact trajectories for almost all, with respect to Lebesgue measure, initial conditions. Interesting questions arise in analysis of space discretizations of systems with SRB invariant measures. Many reasonable computer realizations of such systems can be treated as deterministic mappings φ of some finite subset L (such as computer arithmetic) into itself and we consider only realizations of such type. Such discretizations are also very sensitive to initial conditions and perturbations, but each trajectory of a spatial discretization is eventually periodic and so is not apparently random, as is the case in a continuum. Consequently, the main long-term statistical properties of a discretization are those of its cycles. One of the most important questions in this area concerns the size of the basin of attraction of a typical cycle of a typical discretization for a given chaotic system. An answer to this problem is important in understanding some recently discovered phenomena in the modeling of systems with chaotic behavior (see, for instance, ref. 13). A natural measure of a basin of attraction of a particular cycle is the proportion of points of a lattice which are attracted by this cycle. However, even the mean value of such proportions, averaged over all cycles of a particular discretization, depends on the discretization in a highly irregular and seemingly random way. To circumvent this disconcerting variability, we consider *ensembles* of discretizations and the corresponding statistical *distributions* of characteristics of individual discretizations.

Erber *et al.* (refs. 11 and 12 and references therein) studied the statistics of basins of attraction for various chaotic dynamical systems. Gavelek and Erber[14] observed that, while simulation of such systems is often used to generate statistical information about their average behavior, there are no general results guaranteeing that the computed statistics truly reflect the theoretical average behavior of the system. Lanford[22] has considered the computer statistics from the viewpoint of finite set mappings in computer arithmetic, induced by expansive maps. His experiments showed that the number of attractive cycles was small, that their length was short relative to the size of the lattice, and that virtually all trajectories were attracted to these cycles.

Although these workers have studied some statistical properties of computer simulations and their basins of attraction, systematic models are thin on the ground. In particular, there are no statistical models which provide distributions for a comprehensive number of statistical properties of average behavior of the computer mappings which arise in computation. The gist of this paper is that only a few types of limit distributions can arise provided that the system under consideration has an SRB invariant measure. Moreover, the properties of these distributions can be qualitatively and quantitatively derived from the mathematical analysis of simple phenomenological models.

To obtain results for which the behavior of discretizations may be compared with a developed theory, we commence with a well-understood one-dimensional family of mappings $f_{l,\varepsilon}(x) = (1-\varepsilon)(1-|1-2x|^l)$, $0 \leqslant x \leqslant 1$. The exponent $l \geqslant 1$ and $\varepsilon > 0$ is a small parameter. The statistical properties we study are described in Section 1.1 and our principal results are presented in Section 1.2. They are supported by analysis of a stochastic model of the discretization procedure. This analysis is mathematically rigorous; however, the model itself is, like any phenomenological model, to a certain extent heuristic. It is for this reason that the model is formulated as a hypothesis and why the additional support by numerical experiments is essential. Results of experimental calculations are discussed in Section 1.3. Further, in Section 1.4 we consider briefly discretizations of rotations of the unit circle with an irrational angle. Although for this mapping the Lebesgue measure is an SRB invariant measure, the properties of limit distributions of basins of cycles of discretizations of rotations differ. In Section 2 some two-dimensional systems, such as the Hénon mapping, Lozi mapping, and Anosov systems, are considered. There it is shown that some regularities mentioned for one-dimensional systems are of general nature and are valid for many multidimensional systems with chaotic behavior.

1. ONE-DIMENSIONAL SYSTEMS

1.1. Statistical Properties of Discrete Dynamical Systems

Let L be a given finite set and consider the dynamical system generated by a mapping φ: $\mathrm{L} \to \mathrm{L}$. By $\mathrm{Tr}(\xi_0; \varphi)$ denote the trajectory of φ originating at $\xi \in \mathrm{L}$, that is, $\mathrm{Tr}(\xi_0; \varphi)$ is the sequence $\boldsymbol{\xi} = \xi_0, \xi_1, \ldots, \xi_n, \ldots$ which is defined by $\xi_n = \varphi(\xi_{n-1})$, $n = 1, 2, \ldots$. For a positive integer m the m-shift of a trajectory $\boldsymbol{\xi}$ is the sequence $S^m(\boldsymbol{\xi}) = \xi_m, \xi_{m+1}, \ldots$ and this is also a trajectory of φ. A trajectory $\mathrm{Tr}(\xi_0; \varphi)$ is called a *cycle* if there exists a positive integer N with $\xi_N = \xi_0$. Then $\xi_i = \xi_{i+N}$ for all positive integers i. The minimal N satisfying $\xi_N = \xi_0$ is called the *period* of the cycle. Two cycles either do not contain elements in common or one is a shift of the other. So the totality of cycles is naturally partitioned into a set of equivalence classes. We will call cycles from the same equivalence class *congruent*. Since L is finite, every trajectory $\boldsymbol{\xi}$ of the system is eventually cyclic, that is, there exists a positive integer m such that the shifted trajectory $S^m(\boldsymbol{\xi})$ is a cycle. The minimal m such that $S^m(\boldsymbol{\xi})$ is a cycle is the *length of the transient part* of the trajectory $\boldsymbol{\xi}$ and is denoted by $\mathscr{Q}(\boldsymbol{\xi})$. The sequence $S^{\mathscr{Q}(\boldsymbol{\xi})}(\boldsymbol{\xi})$ is the *cyclic part* of $\boldsymbol{\xi}$. The set L is partitioned into equivalence classes of elements which eventually generate the same cycle. That is, elements ξ_0 and ζ_0 are *equivalent* if the cyclic parts of trajectories $\mathrm{Tr}(\xi_0; \varphi)$ and $\mathrm{Tr}(\zeta_0; \varphi)$ are congruent. Denote by $\mathscr{E}(\varphi)$ the set of such equivalence classes and by $E(\xi; \varphi)$ the equivalence class from $\mathscr{E}(\varphi)$ which contains ξ. Finally, introduce the function $B(x; \varphi)$ defined by

$$B(\xi; \varphi) = \frac{\#E(\xi; \varphi)}{\#(\mathrm{L})}, \qquad \xi \in \mathrm{L} \tag{1.1}$$

Here the symbol $\#(A)$ denotes cardinality of the set A.

Let S denote a finite set of nonnegative real numbers from $[0, 1]$. Define the *distribution function of the set* S, $\mathscr{D}(\cdot; \mathrm{S})\colon [0, 1] \to [0, 1]$ by

$$\mathscr{D}(x; \mathrm{S}) = \frac{\#(\{s \in \mathrm{S} : s \leqslant x\})}{\#(\mathrm{S})}, \qquad 0 \leqslant x \leqslant 1$$

Denote $U(x; \varphi) = \mathscr{D}(x; \{B(\xi; \varphi) : \xi \in L\})$. The function $U(x; \varphi)$ can be interpreted as the distribution in the probability theory sense of the basin of attraction of the cyclic part of a trajectory $\mathrm{Tr}(\xi_0; \varphi)$ with the random initial element ξ_0 uniformly distributed in L. We will be interested in the mean value $V(\varphi)$ of the function $B(\xi, \varphi)$ averaged over $\xi \in \mathrm{L}$. Clearly,

$$V(\varphi) = 1 - \int_0^1 U(x; \varphi)\, dx = \frac{1}{(\#(L))^2} \sum_{E \in \mathscr{E}(\varphi)} (\#(E))^2$$

Thus, $V(\varphi)$ can be interpreted as the probability that two randomly chosen elements of L generate the same cycle. We emphasize that the second statistic is a scalar, while the first one is a scalar function $U(\cdot; \varphi)$ on L.

These concepts can be applied to discretizations of continuous dynamical systems. Let f be a mapping $f\colon [0, 1] \to [0, 1]$ and let ν be a positive integer. Consider the lattice $L_\nu = \{0, 1/\nu, \ldots, (\nu-1)/\nu, 1\}$. The ν-discretization of a mapping $f\colon [0, 1] \mapsto [0, 1]$ is defined by $\varphi(\xi) = [f(\xi)]_\nu$, where $[\alpha]_\nu$ is a scalar roundoff operator, $[\alpha]_\nu = k/\nu$ if $(k - 0.5)/\nu \leqslant \alpha < (k + 0.5)/\nu$, for an integer k. For $\nu = 2^N$ the ν-discretization is a natural theoretical model for implementation of the mapping f in *fixed point format* with N binary digits and radix point in the first position (see, for example, ref. 4, pp. 98–100). If φ is the ν-discretization of f, denote $B(\xi; \varphi)$ by $B_\nu(\xi; f)$, $U(x; \varphi)$ by $U_\nu(x; f)$, and $V(\varphi)$ by $V_\nu(f)$.

Consider the sequences

$$\mathbf{U}(f) = U_1(x; f), U_2(x; f), \ldots, U_\nu(x; f), \ldots \tag{1.2}$$

$$\mathbf{V}(f) = V_1(f), V_2(f), \ldots, V_\nu(f), \ldots \tag{1.3}$$

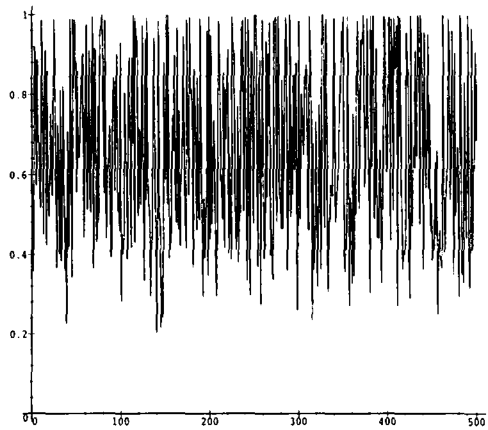

Fig. 1. Mean size $V_{N+n}(f_{l,\varepsilon})$ as function of n for $N=10^6$, $l=2$, $\varepsilon=10^{-3}$.

For large ν, elements of these sequences depend on ν only irregularly when the function f behaves chaotically and the autocorrelation is negligible. For instance, Fig. 1 graphs the elements of the finite sequence

$$V_{N+1}(f_{l,\varepsilon}),..., V_{N+500}(f_{l,\varepsilon})$$

for $l=2$, $\varepsilon=10^{-3}$, $N=10^6$. However, these sequences do have some asymptotic statistical features which can be described. A sequence $\mathrm{s}=s_1, s_2,..., s_\nu,...$ is said to have the *stable distribution property* with a limit $D(x)$ if $\lim_{\nu\to\infty} \mathscr{D}(x; \{s_1, s_2,..., s_\nu\}) = D(x)$. A discussion of stable statistical properties can be found in ref. 20. We will say that the sequence of functions $w_\nu(x)$ is *Cesàro stable with a limit* $D(x)$ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\nu=1}^{N} w_\nu(x) = D(x)$$

Cesàro stability of the sequence (1.2) can be connected with the stable distribution property of a sequence associated with $\{B_\nu(\xi; f)\}$. To this end introduce a random sequence

$$\boldsymbol{\xi} = \xi_1, \xi_2,..., \xi_\nu,...$$

where the ξ_ν are independent and each ξ_ν is uniformly distributed on L_ν and consider the random sequence

$$\boldsymbol{\beta}(f) = B_1(\xi_1; f), B_2(\xi_2; f),..., B_\nu(\xi_\nu; f),... \tag{1.4}$$

Proposition 1. Let the sequence (1.4) have with probability 1 (w.p.1), the stable distribution property with a continuous limit $D(x)$. Then the sequence $\mathbf{U}(f)$ is Cesàro stable with the same limit.

1.2. Main Hypothesis

Let $f_{l,\varepsilon}$ denote the mapping $[0, 1] \mapsto [0, 1]$ which is defined by

$$f_{l,\varepsilon}(x) = (1-\varepsilon)(1 - |1-2x|^l) \tag{1.5}$$

with parameters $l \geqslant 1$ and $\varepsilon > 0$.

Hypothesis 1. (a) Let $1 \leqslant l \leqslant 2$ and let $f_{l,\varepsilon}$ have an absolutely continuous SRB measure with positive density. Then the sequence $\mathbf{U}(f_{l,\varepsilon})$ is Cesàro stable with the limit $1-(1-x)^{1/2}$ and the sequence $\mathbf{V}(f_{l,\varepsilon})$ has the stable distribution property with a continuous limit $D_V(x)$ which does not depend on ε nor on l.

(b) Let $l \geqslant 2$ and let $f_{l,\varepsilon}$ have an absolutely continuous SRB measure with positive density. Then the sequence $\mathbf{U}(f_{l,\varepsilon})$ is Cesàro stable with a continuous limit $D_U(x; l, \varepsilon)$ and the sequence $\mathbf{V}(f_{l,\varepsilon})$ has the stable distribution property with a limit $D_V(x; l, \varepsilon)$.

First of all we extract from this hypothesis a corollary which is more convenient for interpretation and numerical verification. To this end, recall some properties of the set $\tau(l) = \{\varepsilon \in (0, 1) : f_{l,\varepsilon} \text{ has an SRB measure}\}$. The mappings $f_{l,\varepsilon}$ are unimodal mappings with negative Schwartzian,

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 < 0$$

except at the critical point 1/2. Results of ref. 19 show that the mappings $f_{l,\varepsilon}$ have absolutely continuous SRB invariant measures $\mu(l, \varepsilon)$ for some sets $\tau(l) \subset [0, 1)$ and 1 is a density point of each $\tau(l)$, that is, the Lebesgue measure of $\tau(l)$, $\operatorname{mes}(\tau(l))$ is positive and

$$\lim_{\delta \to 0} \frac{\operatorname{mes}([1-\delta, 1] \cap \tau(l))}{\delta} = 1$$

Note that the set $\tau(l)$ is not generic in a topological sense, but is a set of the first Baire category, that is, a closed set without internal points. Hypothesis 1 implies the following assertion.

Corollary 1. For each $l \geqslant 1$ there exists a set $\tau(l) \subset [0, 1)$ which has the number 1 as its density point, such that the following assertions are true.

(a) For $1 \leqslant l \leqslant 2$ and $\varepsilon \in \tau(l)$ the sequence $\mathbf{U}(f_{l,\varepsilon})$ is Cesàro stable with the limit $1-(1-x)^{1/2}$ and the sequence $\mathbf{V}(f_{l,\varepsilon})$ has the stable distribution property with a continuous limit $D_V(x)$ which does not depend on ε nor on l.

(b) For $l \geqslant 2$ and $\varepsilon \in \delta(l)$ the sequence $\mathbf{U}(f_{l,\varepsilon})$ is Cesàro stable with a continuous limit $D_U(x; l, \varepsilon)$ and the sequence $\mathbf{V}(f_{l,\varepsilon})$ has the stable distribution property with a continuous limit $D_V(x; l, \varepsilon)$.

This corollary is strongly supported by results of numerical simulations to be discussed in the next subsection. This gives strong credence to the hypothesis or some very similar supposition. Although not a rigorous justification, there are some physical and heuristic arguments which led to the adoption of this hypothesis and these are set out below. These not only provide a strong motivation, but also give other representations for the limit functions above.

Suppose that we have an ensemble of points $i=0, 1,..., K$ with corresponding masses $\lambda(i)$. Connect each point of the ensemble by an arrow to its image, which may be itself or another point, such that the image of each point is chosen in a random manner *with probability proportional to the mass of the image*. The mapping $T(\cdot \mid \Lambda)$ is the random mapping relating to each point i the endpoint of the arrow originating at i. This is a random graph: arrows between nodes are chosen *randomly*, but once chosen, they are fixed in a realization of the random structure. In particular, a realization of the random mapping is a deterministic discrete dynamical system on the set $X(K) = \{0, 1,..., K\}$. This last property distinguishes random mappings from Markov processes. Formally, the mapping $T_{\Lambda,K}$ is defined by $P(T_{\Lambda,K}(i)=j)=\lambda_j$, where $0 \leqslant i, j \leqslant K$ and the image of an element i is chosen independently of all other images.

In the context of this paper one class of random mappings is distinguished. Let

$$\Lambda_0 = \Delta/(\Delta+K), \qquad \lambda_i = 1/(\Delta+K), \qquad i=1,..., K \tag{1.6}$$

where $\Delta \geqslant 1$ is a parameter. This parameter can be interpreted roughly as the *number* of points which are attracted by the center 0. Then the corresponding random mapping is called a *random mapping with a single attracting center* and is denoted by $T_{\Delta,K}$. In particular, the mapping $T_{1,K}$ is a completely random mapping.[2]

Any realization T of a random mapping $T_{\Lambda,K}$ is a dynamical system on $X(K)$. Therefore the quantity $V(T)$ and the random variable $\beta(i; T)$, where i is uniformly distributed on $X(K)$, are well defined. It is generally recognized that random mappings are good models for analyzing statistical properties of discretizations of systems with an SRB invariant measure.[23, 17] This observation can be formulated using notations introduced in the previous section. First, for $\nu = 1, 2, \ldots$, let Λ_ν, K_ν be sequences of constants, let T_ν be a sequence of independent realizations of T_{Λ_ν, K_ν}, and let i_ν be a sequence of independent random variables uniformly distributed on $X(K_\nu)$. This induces a sequence of random equivalence classes $E(i_\nu; T_\nu)$, each from $X(K_\nu)$, and, associated with this, sequences of random variables

$$\beta(i_\nu; T_\nu) = \frac{\#(E(i_\nu; T_\nu))}{K_\nu + 1}$$

$$v(T_\nu) = \mathbf{E}(\beta(i_\nu; T_\nu) \mid T_\nu)$$

Here, $\mathbf{E}(\omega \mid \zeta)$ denotes the expected value of ω conditioned on ζ.

This observation can be formalized in terms of the statistical properties introduced in the previous section as follows.

Principle of Correspondence. Let a chaotic dynamical system f have an SRB invariant measure. Then for $\nu = 1, 2, \ldots$ there exist constants $K_\nu(f)$, $\Lambda_\nu(f)$ such that the statistical properties of the sequence $\mathbf{V}(f)$ and statistical properties which hold w.p.1 for the random sequence $\boldsymbol{\beta}(f)$ are respectively similar to those which hold w.p.1 for the sequences $V(T_\nu)$ and $\beta(i_\nu; T_\nu)$, $\nu = 1, 2, \ldots$, where T_ν is a sequence of independent realizations of random mappings $T_{\Lambda_\nu(f), K_\nu(f)}$ and i_ν are independent random variables each uniformly distributed in $X(K_\nu)$.

Appropriate $K_\nu(f_{l,\varepsilon})$ and $\Lambda_\nu(f_{l,\varepsilon})$ for $\varepsilon \in \tau(l)$ must be chosen. Arguments from ref. 17 suggest $K_\nu(f_{l,\varepsilon}) = [\nu^{\dim_c(\mu_{l,\varepsilon})}]$, where $\dim_c(\mu)$ is the correlation dimension of μ,[16] and $[\alpha]$ denotes the integer part of α. This leads to the formula

$$K_\nu(f_{l,\varepsilon}) = \begin{cases} \nu & \text{if } 1 < l < 2 \\ [\nu^{2/l}], & \text{if } k > 2 \end{cases}$$

Choice of appropriate $\Lambda_\nu(f_{l,\varepsilon})$ is a little more delicate. It is not enough and not natural to choose $\lambda_i = 1/(K+1)$, $i = 0, \ldots, K$, that is, to consider completely random mappings. Indeed, one point of the lattice L_ν, namely the discretized critical point $[1/2]_\nu$, is not typical because this point immediately attracts a proportion $O(\nu^{1-1/l})$ of other points of the lattice and hence should be given a relative weight in the random mapping model of $O(\nu^{1/l})$. This very fact leads naturally to defining $\Lambda_\nu(f_{l,\varepsilon})$ consistent with (1.6), where $\Delta = \Delta_\nu(f_{l,\varepsilon}) = c\nu^{1/l}$ and $K = K_\nu(f_{l,\varepsilon})$ above. For more accurate but more lengthy reasoning of this kind see ref. 9. Finally, the principle of correspondence above can be reformulated in the following form.

Principle of Correspondence*. Let $\varepsilon \in \tau(l)$. Then there exists a constant $c = c(l, \varepsilon)$ such that the statistical properties of the sequences $\mathbf{V}(f_{l,\varepsilon})$ and statistical properties which hold w.p.1 for the random sequence $\boldsymbol{\beta}(f)$ are similar to those which hold w.p.1 for the sequences $V(T_\nu)$, $\beta(i_\nu; T_\nu)$, $\nu = 1, 2,\dots$, where T_ν is a sequence of independent realizations of random mappings

$$T_{c\nu^{1/l}, [\nu^{\min\{1, 2/l\}}]}$$

This principle is not a rigorous theorem, but more a useful guide for insights into properties of discretizations of dynamical systems. In other words, it can be considered as a phenomenological model of discretizations. To apply this principle it is necessary to know the limit properties of the sequences of random variables $V(T_{\Delta, K})$, $\beta(i_\nu; T_{\Delta, k})$. Much of this information can be found in refs. 3 and 28; see also ref. 2 and references therein. Consider, for instance, some results which are related to the analysis of sequences $\boldsymbol{\beta}(f_{l,\varepsilon})$.

Define the *central component*[3] $A(T_{\Delta, K})$ of $T_{\Delta, K}$ as the random subset of $\{0, 1,\dots, K\}$,

$$A_{\Delta, K} = \{i \in E(K) \colon T^n_{\Delta, K}\, i \in C \text{ for some } n\}$$

Introduce the family of functions

$$D(x; c) = \operatorname{erfc}\left(\frac{c\sqrt{1-x}}{\sqrt{2x}}\right) - e^{c^2/2}(\sqrt{1-x}) \operatorname{erfc}\left(\frac{c}{\sqrt{2x}}\right), \qquad 0 \leqslant x \leqslant 1 \quad (1.7)$$

Here $\operatorname{erfc}(x)$ is the complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt, \qquad x \geqslant 0$$

Clearly, $D(x; c)$ is increasing, with $D(0; c) = 0$, $D(1; c) = 1$, and $D(x; 0) = 1 - (1-x)^{1/2}$.

Proposition 2. (a) Let $\Delta/\sqrt{K} \to 0$ as $K \to \infty$. Then the random variables $a(K) \Leftarrow \#(A(T_{\Delta, K}))$ converge in distribution to $1 - (1-x)^{1/2}$.

(b) Let $\Delta/\sqrt{K} \to c$ as $K \to \infty$. Then the random variables $a(K)$, $K = 1, 2,\dots$, converge in distribution to $D(x; c)$.

Proof. The first assertion follows from Burtin's statement (II), p. 411, ref. 3. The second one follows from the formula (10) in ref. 3, p. 410, by the usual routine. Note that the seemingly more convenient formula in item

(III), p. 411, in the same paper contains a misprint: the denominator of the last integrand should be $\theta_1^{3/2}\theta_4^{3/2}(1-\theta_1-\theta_4)^{1/2}$. ∎

The proposition above and the Central Statistical Theorem (ref. 25, p. 20) imply the following.

Corollary 2. (a) Let $1<l\leqslant 2$. Then the random sequence $\beta(i_\nu, T_\nu)$, where $\{T_\nu\}$ is a sequence of independent realizations of random mappings $\{T_{c\nu^{1/l},\nu}\}$ has w.p.1 the stable distribution property with the limit $1-(1-x)^{1/2}$.

(b) Let $l\geqslant 2$. Then the random sequence $\{\beta(i_\nu, T_\nu)\}$, where $\{T_\nu\}$ is the sequence of independent realizations of random mappings $\{T_{c\nu^{1/l},[\nu^{2/l}]}\}$, has w.p.1 the stable distribution property with the limit

$$D_\beta(x; c)=xD(x; c)+\int_x^1 D(\theta; c)\,d\theta$$
$$-\frac{1}{2}\int_x^1 D(1-\theta; c)\left[\left(\frac{\theta-x}{\theta}\right)^{1/2}+\left(\frac{\theta}{\theta-x}\right)^{1/2}\right]d\theta \qquad (1.8)$$

From the Principle of Correspondence* and Proposition 1, the assertion (a) above leads to the assertion of Hypothesis 1(a) concerning the sequence $\mathbf{U}(f_{l,\varepsilon})$. In the same way, the assertion (b) above leads to the statement of Hypothesis 1(b) about the sequence $\mathbf{U}(f_{l,\varepsilon})$. Moreover, we arrive at the following representation of the functions $D_U(x; l, \varepsilon)$:

$$D_U(x; l, \varepsilon)\approx D_\beta(x; c) \qquad (1.9)$$

where $c=c(l,\varepsilon)$ is a positive parameter chosen for this to hold.

We were unable to find an explicit form for the limit distribution $D_V(x)$, but that part of the Principle of Correspondence* concerning the sequence $\mathbf{V}(f)$ leads to the following formula:

$$D_V(x)\approx \lim_{K\to\infty} D_K(x), \qquad D_V(x; l, \varepsilon)\approx \lim_{K\to\infty} D_K(x; c) \qquad (1.10)$$

Here $D_K(x)$ denotes the distribution of the random variable $V(T_{1,K})$, whereas $D_K(x; c)$ denotes the distribution of $V(T_{c\sqrt{K},K})$ and the parameter c is chosen as in (1.9).

1.3. Numerical Experiments

Case $l\leqslant 2$. The Hypothesis 1(a), with respect to the sequence $\mathbf{U}(f_{l,\varepsilon})$, implies that for $1\ll n\ll N$ the function

$$\mathbf{u}(x; f_{l,\varepsilon}, N, n)=\sum_{\nu=N+1}^{N+n} U_\nu(f_{l,\varepsilon}) \qquad (1.11)$$

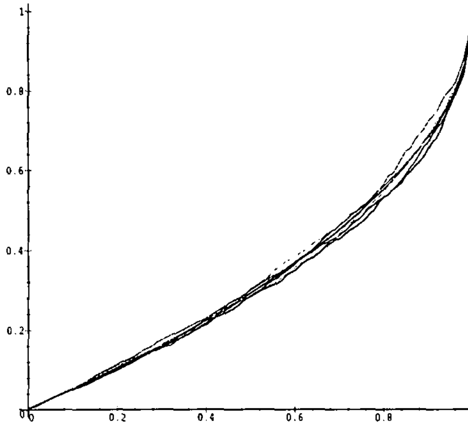

Fig. 2. The function $1-(1-x)^{1/2}$ (the smooth line in the middle) against the functions $\mathbf{u}(x; f_{l,\varepsilon}, N, n)$ for $l=1.0$, 1.2, 1.4, 1.6, 1.8, and 2.0, and $\varepsilon=10^{-3}$, $N=10^6$, $n=10^3$ (jagged lines).

is close to $1-(1-x)^{1/2}$ for $\varepsilon \in \tau(l)$. By Corollary 1(a), the functions $\mathbf{u}(x; f_{l,\varepsilon}, N, n)$ are very nearly $1-(1-x)^{1/2}$ for most sufficiently small $\varepsilon > 0$, in the sense of Lebesgue measure, since 1 is a density point of $\tau(l)$. Recall that the set $\tau(l)$ where Corollary 1 is applicable is thin in the topological sense, because it is a set of the first Baire category. In the context of numerical experiments below, the measure theory properties clearly outweigh the topological properties.

The functions $\mathbf{u}(x; f_{l,\varepsilon}, N, n)$ were calculated for $l=1.0$, 1.2, 1.4, 1.6, 1.8, and 2.0 and $\varepsilon=10^{-3}$, $N=10^6$, $n=10^3$. Results are shown in Fig. 2.

Now consider Hypothesis 1(a) with respect to the sequence $\mathbf{V}(f_{l,\varepsilon})$. For positive integers N and n consider functions

$$\mathbf{v}(x; f_{l,\varepsilon}, N, n) = \mathscr{D}(x; \{V_{N+1}(f_{l,\varepsilon}),..., V_{N+n}(f_{l,\varepsilon})\})$$

The hypothesis with respect to the sequence $\mathbf{V}(f_{l,\varepsilon})$ means that for $1 \ll n \ll N$ and for $\varepsilon \in \tau(l)$, functions $\mathbf{v}(x; f_{l,\varepsilon}, N, n)$ should all be much alike. Moreover, since $D_V(x) \approx D_K(x)$ for large K, all these functions are close to the distribution of the r.v. $V(T_{1,K})$. It is easy to approximate this distribution numerically using a random generator. For an integer K and for $n=1,..., N$, consider the mappings $i \to T_K^{(n)}(i) = a[i, n]$ where the array $a[i, n]$ is generated by SUN Pascal strings

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for n:=1 to N do for i:=0 to K do
begin a[i,n]:=trunc((K+1) * random(i)); end;
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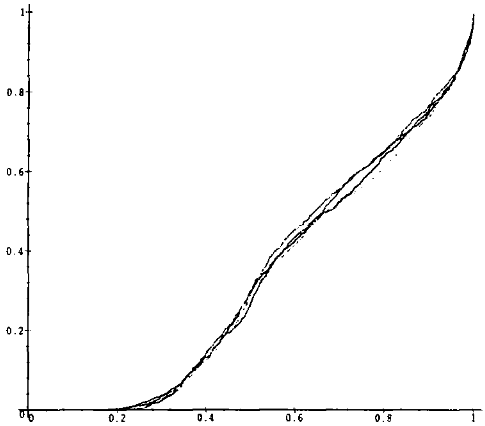

Fig. 3. The distribution $\tilde{\mathscr{D}}^{(0)}(x) = \mathscr{D}(x; \{V(T_K^{(1)}), V(T_K^{(2)}),..., V(T_K^{(10^3)})\})$ for $K = 2^{16} - 1$ and the functions $\mathbf{v}(x, f_{l,\varepsilon}, N, n)$, $l = 1.0$, 1.2, 1.4, 1.6, 1.8, and 2.0, and $\varepsilon = 10^{-3}$, $N = 10^6$, $n = 10^3$.

The formal algorithm is included to indicate the use of a concrete quasirandom generator which can slightly influence the numerical results. Recall the definition of the equivalence class $\mathscr{E}_\varphi$ and for each $n = 1,..., N$, denote

$$V(T_K^{(n)}) = (K+1)^{-2} \sum_{E \in \mathscr{E}(T^{(n)}K)} (\#(E))^2$$

The mappings T_K^n, $n = 1,..., N$, can be considered as N independent sample realizations of the completely random mapping $T_{1,K}$. Therefore, the distribution function of the set $\{V(T_K^{(1)}),..., V(T_K^{(N)})\}$ should be close to the distribution of $V(T_{1,K})$ for reasonably large K, N. Fig. 3 graphs the distribution function

$$\tilde{\mathscr{D}}^{(0)}(x) = \mathscr{D}(x; \{V(T_K^{(1)}), V(T_K^{(2)}),..., V(T_K^{(10^3)})\}) \tag{1.12}$$

for $K = 2^{15} - 1$ and the functions $\mathbf{v}(x; f_{l,\varepsilon}, N, n)$ for $\varepsilon = 10^{-3}$, $N = 10^6$, $n = 10^3$, and the same l as in Fig. 2. The value $K = 2^{15} - 1$ was chosen simply because this is the size of the standard built-in random generator.

Case $l > 2$. Let a specific $l > 2$ be chosen. The hypothesis with respect to the sequence $\mathbf{U}$ means that for each $1 \ll n \ll N$ and for $\varepsilon \in \tau(l)$ the functions $\mathbf{u}(x; f_{l,\varepsilon}, N_1, n)$ and $\mathbf{u}(x; f_{l,\varepsilon}, N_2, n)$ should be similar to one another. Further, both functions $\mathbf{u}(x; f_{l,\varepsilon}, N_1, n)$ and $\mathbf{u}(x; f_{l,\varepsilon}, N_2, n)$ should be close to a function $D(x; c)$ for an appropriate $c = c(l, \varepsilon)$ and both functions $\mathbf{v}(x; f_{l,\varepsilon}, N_1, n)$ and $\mathbf{v}(x; f_{l,\varepsilon}, N_2, n)$ should be close to a function $D_V(x; c)$ for *the same* $c = c(l, \varepsilon)$.

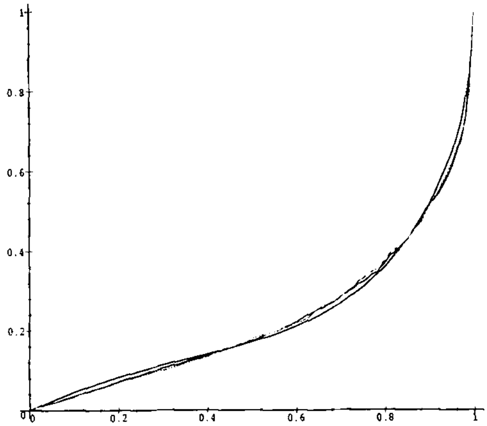

Fig. 4. The functions $\mathbf{u}(x; f_{l,\varepsilon}, 10^5, 10^3)$, $\mathbf{u}(x; f_{l,\varepsilon}, 10^6, 10^3)$, and $D_W^{(1.53)}(x)$.

Consider, for instance $l = 3, \varepsilon = 10^{-3}$. Fig. 4 graphs the functions $\mathbf{u}(x; f_{l,\varepsilon}, 10^5, 10^3)$, $\mathbf{u}(x; f_{l,\varepsilon}, 10^6, 10^3)$, and $D_W(x; 1.53)$. To get a similar figure concerning the second statistic V we need to imitate numerically the function $\lim_{K \to \infty} D_K(x; c)$ for $c = 1.53$. To this end for $n = 1,..., N$, consider the mappings $i \mapsto T_K^{(c,n)}(i) = a[i, n]$, where the array $a[i, n]$ was generated by the SUN Pascal strings

```
for n:=1 to N do for i:=0 to K do
begin a[i,n]:=trunc(random(i) *((K+1)+c * sqrt(K)));
if(a[i,n] >K)then a[i,n]=0; end;
```

For each $n = 1,..., K$, denote

$$V(T_K^{c,n}) = (K+1)^{-2} \sum_{E \in \mathscr{E}(T^{(n)}{}_K)} (\#(E))^2$$

The mappings $T_K^{c,n}$, $n = 1,..., N$, can be considered as N independent sample realizations of the random mapping $T_{c\sqrt{K},K}$. Therefore, the distribution function of the set $\{V(T_K^{c,1}),..., V(T_K^{c,N})\}$ should be close to the distribution of $V(T_{c\sqrt{K},K})$. Fig. 5 graphs $\mathbf{v}(x; f_{l,\varepsilon}, 10^5, 10^3)$, $\mathbf{v}(x; f_{l,\varepsilon}, 10^6, 10^3)$, and the distribution

$$\tilde{\mathscr{D}}^{(1.53)}(x) = \mathscr{D}(x, \{V(T_K^{1.53,1}),..., V(T_K^{1.53,N})\}$$

for $K = 2^{15} - 1$.

Comparable results were obtained for other values of l, such as $l = 2.5$, 3.25, 3.5, 4.0, etc.

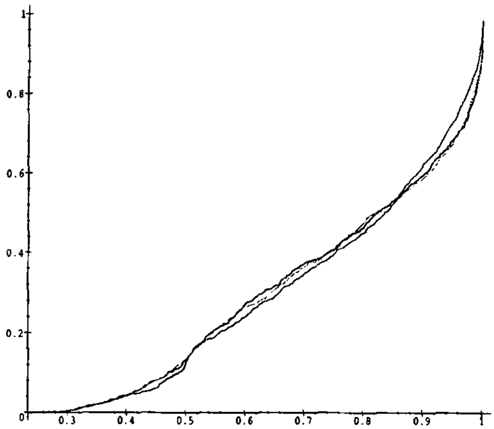

Fig. 5. The functions $\mathbf{v}(x; f_{l,\varepsilon}, 10^5, 10^3)$ and $\mathbf{v}(x; f_{l,\varepsilon}, 10^6, 10^3)$ against the distribution $\tilde{\mathscr{D}}^{1.53}(x) = \mathscr{D}(x; \{V(T_K^{1.53,1}), \ldots, V(T_K^{1.53,N})\}$ for $K = 2^{16} - 1$.

1.4. Rotations of a Circle

Consider the circle $\mathbf{C}$ as the interval $[0, 1]$ with points 0 and 1 identified. Then rotation of the circle by the angle $2\pi\gamma$ in the positive direction is interpreted as the angle mapping $\Psi_\gamma : [0, 1] \to [0, 1]$ defined by $x \mapsto (x + \gamma) \bmod 1$. For each positive integer ν, denote by $\mathbf{L}_\nu^*$ the lattice $\mathbf{L}_\nu$

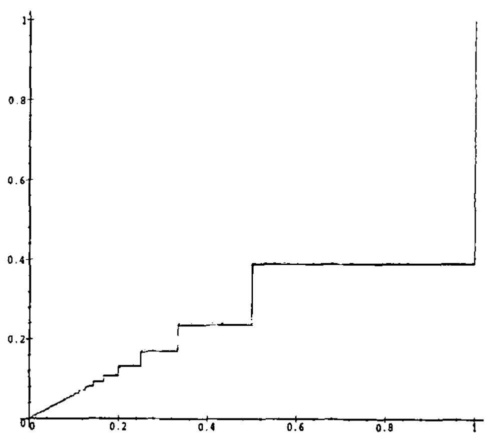

Fig. 6. The function $(6/\pi^2) \sum_{n \geqslant 1/x} (1/n^2)$ against $\mathbf{u}(x; \Psi_{\ln(2)}, 10^5, 10^3)$.

on $[0, 1]$ with 0 and 1 identified and define the corresponding discretization $\psi(\xi) = (\xi + [\gamma]_\nu) \bmod 1$, $\xi \in L_\nu^*$. Define

$$D_*(x) = \frac{6}{\pi^2} \sum_{n \geqslant 1/x} \frac{1}{n^2}$$

Proposition 3. Let γ be irrational. Then the sequence $\mathbf{U}(\Psi_\gamma)$ is Cesàro stable with the limit $D_*(x)$ and the sequence $\mathbf{V}(\Psi_\gamma)$ coincides with $\mathbf{U}(\Psi_\gamma)$.

The proof follows from the definitions. ▌

Figure 6 compares $D_*(x)$ with the function $\mathbf{u}(x; \Psi_{\sqrt{2}}, 10^5, 10^3)$.

2. TWO-DIMENSIONAL SYSTEMS

2.1. Definitions

Consider mappings $f: \mathbb{R}^d \to \mathbb{R}^d$. Let us define the class of computer realizations which we will study. Consider the lattice $L_\nu^d = \nu^{-1}\mathbf{Z}^d$, where $\mathbf{Z}^d$ is the standard integer lattice in $\mathbb{R}^d$. The L_ν^d-discretization f_ν of f is defined by $f_\nu(\xi) = ([y^{(1)}]_\nu, ..., [y^{(d)}]_\nu)$, where $\xi = (\xi^{(1)}, ..., \xi^{(d)}) \in L_\nu^d$, $y = f(\xi) \in \mathbb{R}^d$, and $[\cdot]_\nu$ is the scalar roundoff operator defined above. Write $Q^{z,\rho}$, $z \in \mathbb{R}^d$, $\rho > 0$, for the cube

$$\{x = (x^{(1)}, x^{(2)}, ..., x^{(d)}) \in \mathbb{R}^d\colon |x^{(i)} - z^{(i)}| \leqslant \rho, i = 1, ..., d\}$$

We will consider trajectories of discretizations originating in $Q^{z,\rho}$. Throughout, a fixed vector $z \in \mathbb{R}^d$ and a fixed number $\rho > 0$ will be chosen so that all trajectories ξ satisfying $\xi_0 \in Q^{z,\rho}$ are bounded in L_ν^d and therefore eventually periodic. As in Section 1.1, there is a natural partition of the set $Q_\nu^{z,\rho} = Q^{z,\rho} \cap L_\nu^d$ into the set $\mathscr{E}_\nu^{z,\rho}(f)$ of equivalence classes. Computational statistics of the sequences (1.2) and (1.3) are as follows. For $\nu = 1, 2, ...,$ define

$$U_\nu^{z,\rho}(x; f) = \mathscr{D}(x, \{\#(E)(\#Q_\nu^{z,\rho})^{-1} : E \in \mathscr{E}_\nu^{z,\rho}(f)\})$$

$$V_\nu^{z,\rho}(f) = (\#(Q_\nu^{z,\rho}))^{-2} \sum_{E \in \mathscr{E}_\nu^{z,\rho}(f)} (\#(E))^2$$

Then define the corresponding sequences

$$\mathbf{U}^{z,\rho}(x; f) = \{U_\nu^{z,\rho}(x; f)\}, \qquad \mathbf{V}^{z,\rho}(f) = \{V_\nu^{z,\rho}(f)\}$$

2.2. Properties of Some Chaotic Systems

Several well-known mappings of $\mathbb{R}^2$ will be considered. To avoid cumbersome notations we will denote elements of $\mathbb{R}^2$ by $\mathbf{x} = (x, y)$ and will write $\mathbf{0} = (0, 0)$. Suppose that f has an SRB invariant measure μ. The maximal open set Ω for which μ is a weak limit of the sequence of measures $\mu_n = (1/n) \sum_{i=0}^{n-1} f^i_* \delta_x$ for almost all initial conditions $x \in \Omega$ with respect to Lebesgue measure is the basin of attraction of the measure μ. Also, although a great many chaotic systems have SRB invariant measures, it is not easy to verify such a property for a measure and it is even more difficult to estimate its basin of attraction. In practical situations SRB measures can collapse under discretization.[5,6] Suppression of such collapsing effects is achieved by injecting random or multivalued perturbations into the computation or using different regularization methods (see references in refs. 6 and 7).

Recall the Hénon and Lozi mappings, which are defined, respectively, by

$$f_H(\mathbf{x}; \mathbf{a}) = (1 + y - ax^2, bx), \qquad f_L(\mathbf{x}; \mathbf{a}) = (1 + y - a|x|, bx)$$

where $\mathbf{a} = (a, b)$ is a vector of real parameters. It is thought that the Hénon mapping has an SRB measure μ for some $\mathbf{a}$, while the Lozi mapping certainly has an SRB measure μ for some $\mathbf{a}$, which was rigorously proved in ref. 24.

Hypothesis 2. For some $\mathbf{a}$, suppose that the Hénon mapping (respectively, the Lozi mapping) has an SRB invariant measure μ and the cube $Q^{z,\rho}$ belongs to the basin of attraction of μ. Then the sequence $\mathbf{U}^{z,\rho}(\cdot; f_H(\cdot; \mathbf{a}))$ [respectively $\mathbf{U}^{z,\rho}(\cdot; f_L(\cdot; \mathbf{a}))$] is Cesàro stable with the limit $1-(1-x)^{1/2}$ and the sequence $\mathbf{V}^{z,\rho}(\cdot; f_H(\cdot; \mathbf{a}))$ [respectively $\mathbf{V}^{z,\rho}(\cdot; f_L(\cdot; \mathbf{a}))$] has the stable distribution property with limit $D_V(x) = \lim_{K \to \infty} D_K$.

Numerical experiments have been carried out to test this hypothesis. Let $z = (-0.05, -0.05)$ and $\rho = 0.2$. Fig. 7 compares the functions

$$\mathbf{u}^{z,\rho}(x; f_H(\cdot; 1.4, 0.3), 500, 500), \qquad \mathbf{u}^{z,\rho}(x; f_L(\cdot; 1.7, 0.5), 500, 500)$$

against the function $1-(1-x)^{1/2}$. Analogously, Fig. 8 graphs the functions

$$\mathbf{v}^{z,\rho}(x; f_H(\cdot; 1.4, 0.3), 500, 500), \qquad \mathbf{v}^{z,\rho}(x; f_L(\cdot; 1.7, 0.5), 500, 500)$$

against the function $\bar{\mathscr{D}}^{(0)}(x)$, which was defined in (1.12). The concrete values of parameters a, b were taken from ref. 18, p. 269 and from ref. 19, p. 203.

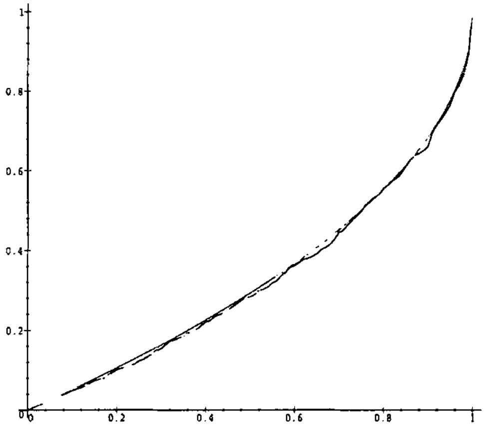

Fig. 7. The functions $\mathbf{u}^{z,\rho}(x; f_H(\cdot; 1.4, 0.3, N, n))$ and $\mathbf{u}^{z,\rho}(x; f_L(\cdot; 1.7, 0.5, N, n))$ (for the Hénon and Lozi mappings) for $z = (-0.05, -0.05)$, $\rho = 0.2$, and $N = n = 500$ against the function $1 - (1 - x)^{1/2}$.

Other combinations of the parameters were tried for the Hénon and Lozi mappings. All experimental results strongly support the hypothesis above. Quite similar results were also obtained for other mappings, such as the Belykh mapping defined by

$$f_{\mathrm{B}}(\mathbf{x}; \lambda_1, \lambda_2, k) = \begin{cases} (\lambda_1((x) - 1) + 1, \lambda_2((y) - 1) + 1 & \text{if} \quad y > kx \\ (\lambda_1((x) + 1) - 1, \lambda_2((y) + 1) - 1 & \text{if} \quad y < kx \end{cases}$$

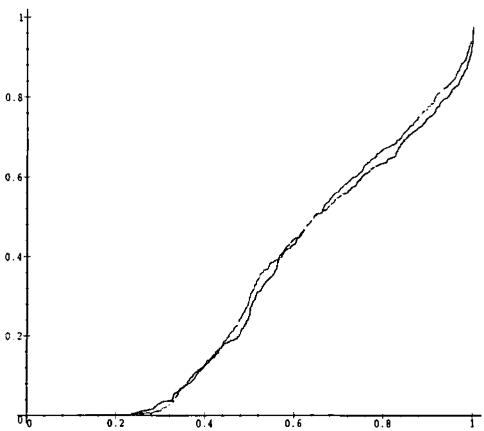

Fig. 8. The functions $\mathbf{v}^{z,\rho}(x; f_H(\cdot; 1.4, 0.3, N, n))$ and $\mathbf{v}^{z,\rho}(x; f_L(\cdot; 1.7, 0.5, N, n))$ for $z = (-0.05, -0.05)$, $\rho = 0.2$, and $N = n = 500$ against the function (1.12).

Here $\lambda_1 \in (0, 1/2)$, $\lambda_2 \in (1, 2)$, $k \in (-1, 1)$ are parameters and (α) is the fractional part of α. This mapping arises in phase synchronization systems. It is especially interesting because it cannot be reduced in any sense to one-dimensional mappings. See further references in ref. 19, p. 203.

In contrast, consider the shift mapping on a two-dimensional torus defined by $g(x, y) = ((x+a), (y+b))$. This mapping also has SRB invariant measure which coincides with Lebesgue measure, provided that a, b are rationally independent. Nevertheless, the cardinality of the basin of attraction in L_ν^2 of each cycle of the corresponding ν-discretization clearly does not exceed ν. In particular, the sequence $\mathbf{U}(g)$ is Cesàro stable with a degenerate limit δ_0, the Dirac distribution at the point 0. Similarly, $\mathbf{V}(g)$ has the stable distribution property with the limit δ_0. Much the same behavior was observed in experiments for discretizations of an algebraic automorphism f_A of the standard 2-torus generated by an integer hyperbolic 2-matrix A with $|\det(A)| = 1$.

Why should circle rotations, toroidal shifts, and algebraic automorphisms of the torus behave so differently from the other mappings with an SRB measure considered above? We believe that a reasonable explanation is that for all these mappings the discretizations have their own algebraic structure. This additional structure precludes the possibility of using random mappings as models of discretizations of the original mapping. In particular, the discretizations of these mappings are invertible, and consequently the whole lattice is partitioned into disjoint cycles of the mapping. Hence, such mappings contain many more cycles than do mappings without algebraic structures on discretizations and the basin of attraction of each cycle is much smaller.

2.3. Well-Spread Measures

The results discussed above and other numerical experiments with modifications of horseshoe and twisted horseshoe mappings and β-mappings and its two-dimensional analogs, as well as heuristic reasoning, suggest the following conjecture involving the idea of a well-spread measure.

Conjecture. Suppose that f has an SRB invariant measure μ with positive correlation dimension and let a cube $Q^{\Xi,\rho}$ belong to its basin of attraction. Then the sequence $\mathbf{U}^{\Xi,\rho}(f)$ is Cesàro stable with the limit $1-(1-x)^{1/2}$ and the sequence $\mathbf{V}^{\Xi,\rho}(f)$ has the stable distribution property with the limit $D_V(x) = \lim_{K\to\infty} D_K$ provided that the measure μ is well spread on its support $\mathrm{Supp}(\mu)$ and provided that typical discretizations do not have a strong algebraic structure.

The notion of a well-spread measure is described as follows. Let $\varepsilon > 0$ be fixed and write $Y = \mathrm{Supp}(\mu)$. A finite subset $Z = \{x_1, ..., x_n\} \subseteq Y$ is called ε-separated if $|x_i - x_j| \geqslant \varepsilon$ e for all $i \neq j$, $1 \leqslant i, j \leqslant n$. For any closed subset $A \subseteq Y$ denote by $s_\varepsilon(A)$ the totality of ε-separated subsets $Z \subseteq A$. Denote by $C_\varepsilon(A)$ the binary logarithm of the maximal cardinality of subsets from $s_\varepsilon(A)$. This is called the *ε-capacity* of A. The upper entropy index of Y is the value

$$\dim_f^u(A) = \lim_{\varepsilon \to 0} \sup_{0 < \varepsilon_1 < \varepsilon} \frac{C_{\varepsilon_1}(A)}{|\log_2(\varepsilon_1)|} \tag{2.1}$$

For a fuller description, see ref. 10. Recall also the definition of correlation dimension of a probability measure.[16] Suppose that μ is a probability measure with a compact support $Y = \mathrm{Supp}(\mu)$. This measure induces a natural probability measure on $Y \times Y$ which is denoted by μ^2. For each $\varepsilon > 0$ let $T(\varepsilon, Y)$ be a subset of $Y \times Y$ which includes all pairs (x, y) with $|x - y| \leqslant \varepsilon$. Introduce the quantity $C_\varepsilon(Y, \mu)$ as the absolute value of the binary logarithm of $\mu(T(\varepsilon, Y))$. The *correlation dimension* of μ is the value

$$\dim_c^u(\mu) = \lim_{\varepsilon \to 0} \sup_{0 < \varepsilon_1 < \varepsilon} \frac{C_{\varepsilon_1}(Y, \mu)}{|\log_2(\varepsilon_1)|} \tag{2.2}$$

Proposition 4. $\dim_f^u(Y) = \max_{\bar\mu} \dim_c^u(\bar\mu)$, $\mathrm{Supp}(\bar\mu) \subseteq Y$.

A proof is presented in the next subsection.

Definition. The measure μ is said to be *well spread* on Y if it satisfies Proposition 4.

All examples which have been considered suggest that this definition is adequate for the conjecture above. If an SRB measure is not well spread, then the formulas (1.9) and (1.10) are applicable to the analysis of the basin of attraction of discretizations.

2.4. Proof of Proposition 4

First, to establish the inequality $\dim_c^u(y) \leqslant \dim_f^u(Y)$, it suffices to establish the inequality

$$|\log_2(\mu(T_{2\varepsilon}(Y)))| \leqslant \log_2(\mathcal{N}_\varepsilon(Y)) \tag{2.3}$$

for each μ, where $\mathcal{N}_\varepsilon(Y)$ is the maximal cardinality of subsets from $s_\varepsilon(A)$. This inequality is the same as

$$\mu(T_{2\varepsilon}(Y)) \geqslant \mathcal{N}_\varepsilon(Y)^{-1} \tag{2.4}$$

Let $S=\{x_1,..., x_{\mathcal{N}_\varepsilon(Y)}\}$ be an ε-separated set with maximal cardinality. Denote by $B_\varepsilon(x)$ the closed ball of radius ε centered at x. By definition, $Y=\bigcup_{n=1}^{\mathcal{N}_\varepsilon(Y)} B_\varepsilon(x_n)$. Consequently,

$$T_{2\varepsilon}(Y)\subseteq \bigcup_{n=1}^{\mathcal{N}_\varepsilon(Y)} B_{2\varepsilon}(x_n)\times B_{2\varepsilon}(x_n)$$

and $T_{2\varepsilon}(Y)\supseteq\bigcup_{n=1}^{\mathcal{N}_\varepsilon(Y)} F_n\times F_n$, where $F_n=B_\varepsilon(x_n)\bigcup_{i=1}^{n-1} B_\varepsilon(x_i))$. That is

$$\mu(T_{2\varepsilon}(Y))\geqslant\mu\left(\bigcup_{n=1}^{\mathcal{N}_\varepsilon(Y)} F_n\times F_n\right) \tag{2.5}$$

On the other hand, by construction,

$$Y=\bigcup_{n=1}^{\mathcal{N}_\varepsilon(Y)} F_n \tag{2.6}$$

and $F_n\cap F_m=\varnothing$ for $n\neq m$. Thus

$$\mu\left(\bigcup_{n=1}^{\mathcal{N}_\varepsilon(Y)} F_n\times F_n\right)=\sum_{n=1}^{\mathcal{N}_\varepsilon(Y)}\mu^2(\mathscr{F}_n) \tag{2.7}$$

Now, (2.6) implies that $\sum_{n=1}^{\mathcal{N}_\varepsilon(Y)}\mu(\mathscr{F}_n)=1$. From this and (2.7) it follows that $\mu(\bigcup_{n=1}^{\mathcal{N}_\varepsilon(Y)} F_n\times F_n)\geqslant\mathcal{N}_\varepsilon(Y)^{-1}$. The inequality (2.4) follows from the last inequality and (2.5). The inequality (2.3) is now proven. To finish the proof it remains to construct a measure μ satisfying $\dim_c''(\mu)=\dim_f''(Y)$. It can be chosen, for instance, in the form $\mu=\sum_{\nu=1}^{\infty} 2^{-\nu}\mu_{A(\nu)}$, where $\mu_{A(\nu)}$ is the probability measure distributed uniformly on a suitable finite set $A(\nu)$. The calculation is straightforward and is omitted. ▮

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